Lecture 2B: Graph Theory II

UC Berkeley EECS 70
Summer 2022
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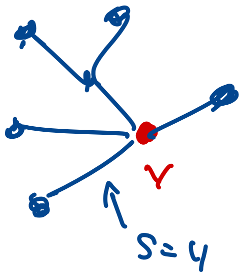
Announcements!

- Read the Weekly Post required recording
- We have caught academic misconduct cases
- **HW 2** and **Vitamin 2** have been released, due **Thu** (grace period Fri)
- Throughout this lecture **<u>definitions</u>** will be underlined

- OH was yday

Minimum Edges for Connectivity



Theorem: Any connected graph with n vertices must have at least n-1 edges

Proceed by strong induction on $n = |V|$

Base Case: $n=1$ $\quad (1)-1 = 0$ ✓

Ind Hyp: Assume claim holds $1 \le n \le k$

Ind Step: Consider an arbitrary graph G with $n = k+1 = |V|$ $\quad G=(V,E)$

Then remove any vertex v, call the resulting graph $G' = (V', E')$

Removing v creates at most deg v connected components; let $s \le \deg v$

Denote k_i to be the number of vertices in the ith connected component

edges in 1st connected component

follows from ind. hyp.

$$|E_1| + |E_2| + \dots + |E_s| \ge (k_1 - 1) + \dots + (k_s - 1)$$

$$|E'| = |E_1| + \dots + |E_s| \ge (k_1 + \dots + k_s) - s$$

we add back

$$|E| = |E'| + s$$

$$|E| = |E'| + \deg v \ge k - s + s$$

vertices

$$(k+1) - 1 = k$$

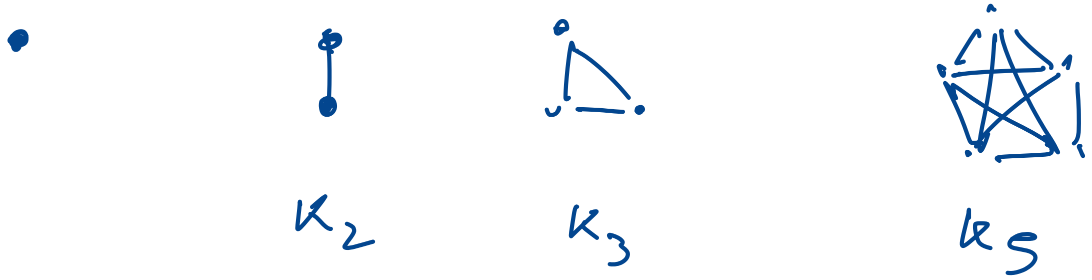
$$|E| \ge k \quad \text{as desired}$$

Complete Graphs

A graph G is **complete** if it contains the maximum number of edges possible.

Correction: K is for mathematician Kazimierz Kuratowski

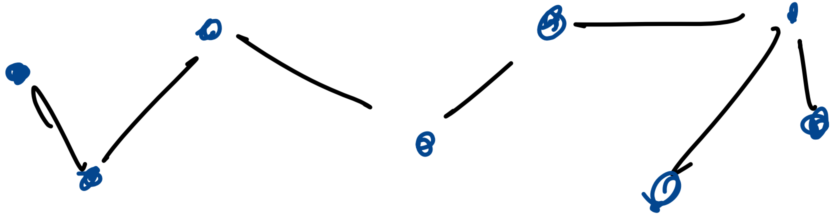
Examples:



Trees

The following definitions are all equivalent to show that a graph G is a **tree**.

1. G is connected and contains no cycles
2. G is connected and has n-1 edges (where $n = |V|$)
3. G is connected, and the remove of any single edge disconnects G
4. G has no cycles, and the addition of any single edge creates a cycle



Tree Definitions are Equivalent

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.
Proof:

$\Rightarrow$ if no cycles, then $n-1$ edges. Proceed by induction on # of vertices

Base case: $n=1$ has no edges ✓

Ind. Hyp: Assume the claim for all $1 \le n \le k$

Ind. Step: Consider a graph with $n+1$ vertices. Remove any arbitrary vertex v.

Case 1: G' is disconnected. Apply ind. hyp to each connected component (similar proof as before)

Case 2: G' is connected, then G' has $n-1$ edges by ind. hyp. We add back v. V can only be incident on one edge otherwise given G' is connected, G must have had a cycle. Thus adding back 1 edge gives k edges for $k+1$ vertices $\square$

Tree Definitions are Equivalent (cont.)

Theorem: For a connected graph G it contains no cycles iff it has n-1 edges.

$\Leftarrow$ if $n-1$ edges, then no cycles

Assume for contradiction G is connected and has $n-1$ edges but also contains a cycle. Then removing an edge from the cycle in G does not disconnect G. Now G' has n vertices but only $n-2$ edges. Which we proved earlier is not possible!

G has no cycles.

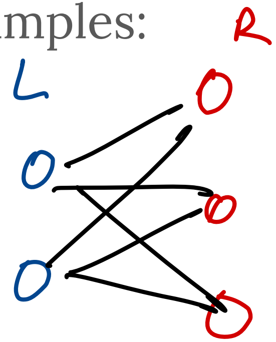
Bipartite Graphs

A graph G is **bipartite** if the vertices can be split in two groups (L or R) and edges only go between groups.

two adjacent vertices don't have the same color
We use at most 2 colors

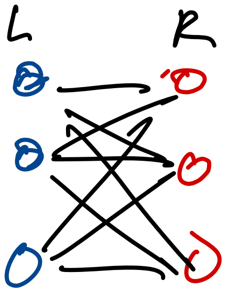
G is bipartite iff G is two colorable

Examples:

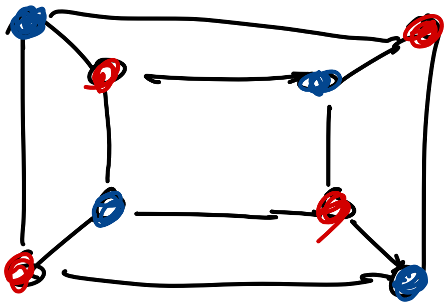


$K_{2,3}$

$K_{3,2}$



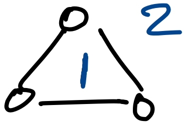
$K_{3,3}$



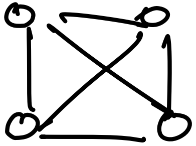
Planar Graphs

A graph is called **planar** if it can be drawn in the plane without any edges crossing.

Examples:

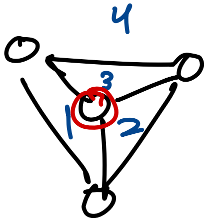


Yes



Yes

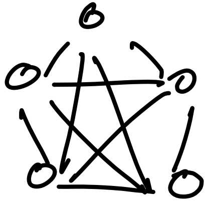
$\equiv$



yes

K_5

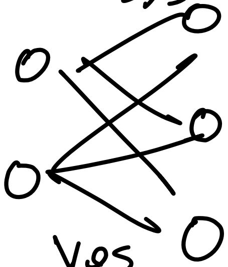
Complete graph



No

Bipartite

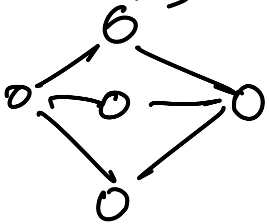
$K_{2,3} \equiv K_{3,2}$



Yes

$\equiv$

$K_{2,3}$



yes

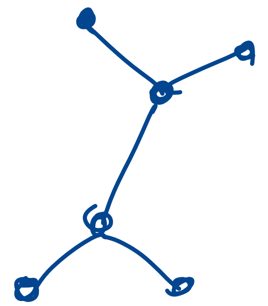
K_{33}

Non planar

Euler's Formula: $v - e + f = 2$

Theorem: If G is a connected planar graph, then $v - e + f = 2$.

Proof: Proceed by induction on e



Base Case: $e=0$, $f=1$, $v=1$ $=$ $1-0+1=2$ ✓

Ind. Hyp: Assume claim holds for $e=k$

Ind. Step: Consider an arbitrary graph G with $e=k+1$

Case 1: G is a tree. Then $f=1$, $v=e+1$

$$(e+1) - e + 1 = 2 \checkmark$$

Case 2: G is not a tree, then G has a cycle. Remove an edge from the cycle. G is still connected and planar. Thus apply ind. hyp.

$$v - e' + f' = 2$$

Adding back the edge creates 1 face

$$v - (e'+1) + (f'+1) = v - e + f = 2 \checkmark$$

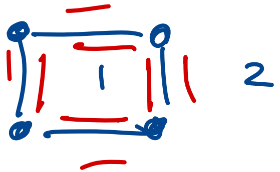
Euler's Formula Corollary: $e \leq 3v - 6$

Corollary: For a connected planar graph with $v \geq 3$, we have $e \leq 3v - 6$

Proof:

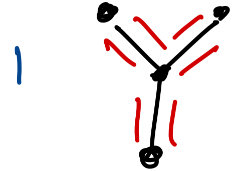
Define a side to be the "side" of an edge towards a face.

Ex:



face 1 has 4 sides

face 2 has 4 sides



sides

Let $s_i :=$ number of sides for ith face

$$\sum_{i=1}^{f} s_i = 2e$$

Each face has 3 sides at least

$$2e = \sum_{i=1}^{f} s_i \geq \sum_{i=1}^{f} 3 \implies 2e \geq 3f$$

$$2e \geq 3(2+e-v) \implies e \leq 3v - 6 \checkmark$$

$$2e \geq 6 + 3e - 3v$$

K_5 is non-planar

Proof:

Assume for contradiction K_5 is planar

$v = 5$

$e = 5 \cdot 4 \cdot \frac{1}{2} = 10$

$\binom{5}{2}$

$e \le 3v - 6$

$10 \le 3(5) - 6$

$10 \not\le 9$

D!

$K_{3,3}$ is non-planar

this is on the HW

Proof:

Assume for contradiction

$v = 6$

$e = 9$

$9 \leq 3v - 6$

$9 \leq 3(6) - 6$

$9 \leq 12$???

Hint: Can you get a stronger inequality than $e \leq 3v - 6$ because $K_{3,3}$ is bipartite

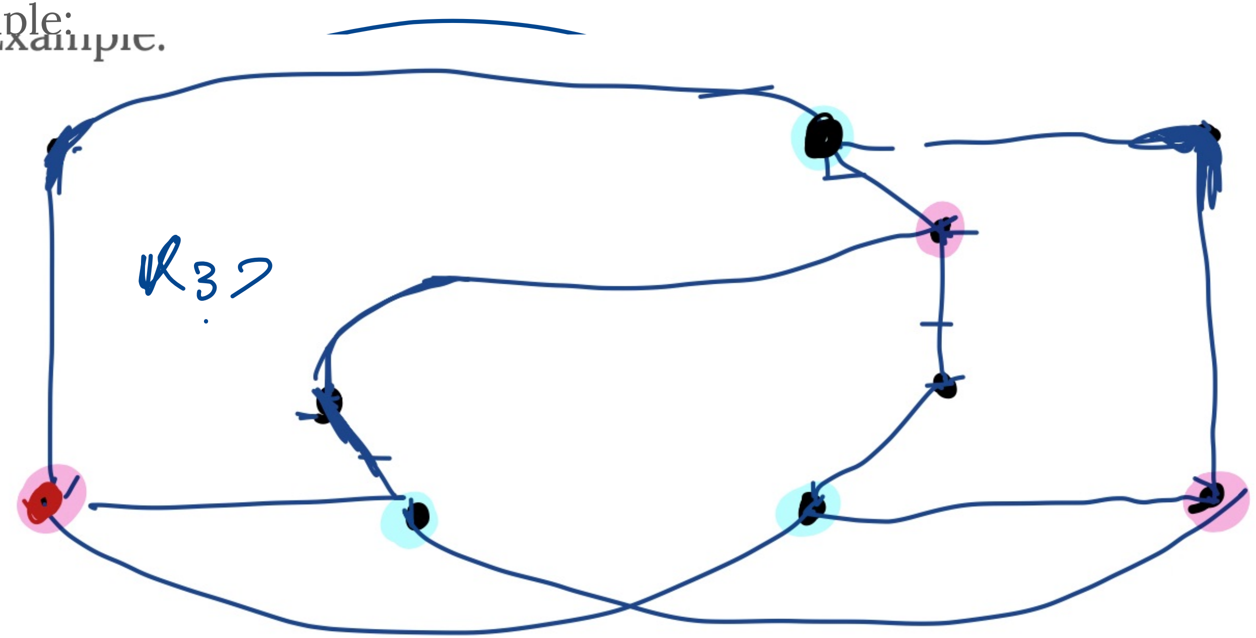
Kuratowski's Theorem

Theorem: A graph is **non-planar** iff it contains K_5 or $K_{3,3}$

Example:

4D hypercube

non-planar

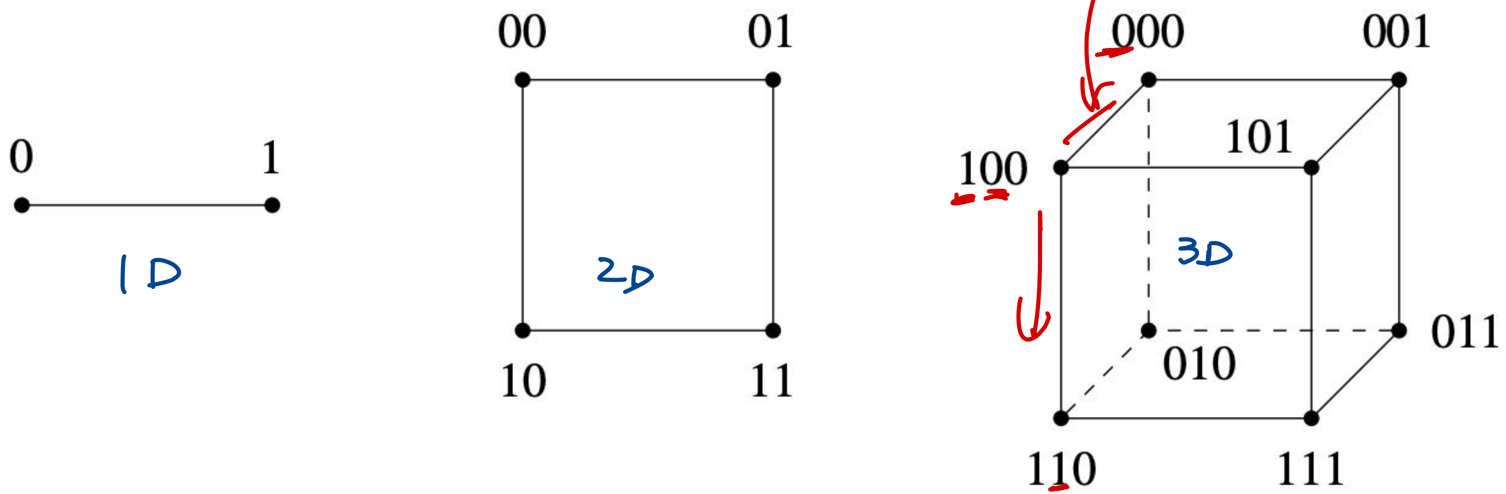


Hypercubes

The vertex set of a n-dimensional **hypercube** G=(V, E) is given by V = $\{0, 1\}^n$ i.e. the vertices are n-bit strings.

n dim $\rightarrow$ 2^n vertices

traversal $\rightarrow$ you flip a 1.



Prove every hypercube is bipartite

Number of Edges in Hypercubes

Lemma: The total number of edges in an n-dimensional hypercube is $n2^{n-1}$

Proof:

$$\sum_{v} \deg(v) = \sum_{v} n = 2^n \cdot n = 2|E|$$

$$|E| = \frac{2^n \cdot n}{2}$$

$$= n2^{n-1}$$